Diagram method in research on coadjoint orbits

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Abstract

We correspond to any factor algebra of the unitriangular Lie algebra with respect to a regular ideal some permutation. In terms of this permutation one can construct a diagram, that allows to calculate index and maximal dimension of coadjoint representation.

0. Introduction

Coadjoint orbits play an important role in representation theory, symplectic geometry, mathematical physics. According to the orbit method of A.A.Kirillov [1, 2] for nilpotent Lie groups there exists one to one correspondence between coadjoint orbits and irreducible representations in Hilbert spaces. This gives a possibility to solve the problems of representation theory and harmonic analysis in geometrical terms of the orbit space. From the other point of view, coadjoint orbits are symplectic manifolds and many of hamiltonian systems of the classical mechanics may be realized on these orbits [4, 5, 6]. However the problem of classification of all coadjoint orbits for specific Lie groups (such as the group of unitriangular matrices) is an open problem that is still interesting [3]. In the origin paper on the orbit method [2] the description of algebra of invariants and classification of orbits of maximal dimension was obtained.

In the paper [7] all coadjoint orbits for the groups $\mathrm{UT}(n, K)$ of the size less or equal to seven were classified. In the same paper we have got a classification of all subregular coadjoint orbits for an arbitrary n.

In the paper [8] we consider the families of coadjoint orbits associated with involutions. The special case is a family of orbits of maximal dimension that is associated with the involution of maximal length. We obtain a formula of dimension of these orbits and construct generators of the defining ideal of the orbit; for the canonical forms in these orbits we construct a polarization.

These paper is a continuation of the paper [9] in which for any Lie algebra $\mathcal{L}$, defined at the end of introduction, we constructed the diagram $\mathcal{D}_{\mathcal{L}}$. By this

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diagram one can easily calculate the maximal dimension of coadjoint orbit and the index of $\mathcal{L}$.

Recall the index of a Lie algebra is a minimal dimension of the stabilizer of a linear form on this Lie algebra. For nilpotent Lie algebras the field of invariants of the coadjoint representation is a pure transcendental extension of the main field with the degree equal to the index of this Lie algebra [10].

In [9] the diagram is constructed as a result of induction procedure which is convenient in special cases and is not convenient for proof of general theorems. One should like to obtain some formula that shows what symbols took place in the diagram.

This problem is solved in this paper. To each considered Lie algebra $\mathcal{L}$ we correspond a permutation $w_{\mathcal{L}}$. The main results are formulated in terms of decompositions of $w_{\mathcal{L}}$ into a product of reflections and its action in the segment $[1, n]$ of positive integers (see theorems 2.2, 2.6 and 2.7). In the sequel of the paper we formulate a conjecture on structure of field of invariants of the coadjoint representation of $\mathcal{L}$.

Turn to presentation of the content of paper. Let $N = \mathrm{UT}(n, K)$ be the group of unitriangular matrices of size $n \times n$ with units on the diagonal and with entries in the field K of zero characteristic. The Lie algebra $\mathfrak{n} = \mathfrak{ut}(n, K)$ of this group consists of lower triangular matrices of size $n \times n$ with zeros on the diagonal. One can define the natural representation of the group N in the conjugate space $\mathfrak{n}^*$ by the formular $\mathrm{Ad}_g^* f(x) = f(\mathrm{Ad}_g^{-1} x)$, where $f \in \mathfrak{n}^*$, $x \in \mathfrak{n}$ and $g \in N$. This representation is called a coadjoint representation. We identify the symmetric algebra $S(\mathfrak{n})$ with the algebra of regular functions $K[\mathfrak{n}^*]$ on the conjugate space $\mathfrak{n}^*$. Let us also identify $\mathfrak{n}^*$ with the subspace of upper triangulat matrices with zeros on the diagonal. The pairing $\mathfrak{n}$ and $\mathfrak{n}^*$ is realized due to the Killing form $(a, b) = \mathrm{Tr}(ab)$, where $a \in \mathfrak{n}$, $b \in \mathfrak{n}^*$. After this identification the coadjoint action may be realized by the formula $\mathrm{Ad}_g^* b = P(\mathrm{Ad}_g b)$, where P is the natural projection of the space of $n \times n$-matrices onto $\mathfrak{n}^*$.

Recall that for any Lie algebra $\mathfrak{g}$ the algebra $K[\mathfrak{g}^*]$ is a Poisson algebra with respect to the Poissson bracket such that $\{x, y\} = [x, y]$ for any $x, y \in \mathfrak{g}$. In the case $k = \mathbb{R}$ the symplectic leaves with respect to this Poisson bracket coincides with the orbits of coadjoint representation [1]. Respectively, the algebra of Casimir elements in $K[\mathfrak{g}^*]$ coincides with the algebra of invariants $K[\mathfrak{g}^*]^N$ of the coadjoint representation.

The coadjoint orbits of the group N are closed with respect to Zariski topology in $\mathfrak{n}^*$, since all orbits of a regular action of an arbitrary algebraic unipotent group in an affine algebraic variety are closed [10, 11.2.4].

To simplify language we shall give the following definition: a root is an arbitrary pair (i,j), where i,j are positive integers from 1 to n and $i \neq j$. The permutation group S_n acts on the set of roots by the formula $w(i,j) = (w(i), w(j))$.

A root (i,j) is positive if $i > j$. Respectively, a root is negative if $i < j$. We denote the set set positive roots by Δ_+.

For any root $\eta = (i,j)$ we denote by $-\eta$ the root (j,i). We define the partial operation of addition on the set of positive roots: if $\eta = (i,j) \in \Delta_+$ and $\eta' = (j,m) \in \Delta_+$, then $\eta + \eta' = (i,m)$.

Consider the standard basis $\{y_{ij} : \ (i,j) \in \Delta_+\}$ in the algebra $\mathfrak{n}$. We shall also use the notation y_ξ for y_{ij}, where $\xi = (i,j)$.

Fix some subset $M \subset \Delta_+$, that satisfies the following condition: if in a sum of two positive roots one of summands belongs to M, then the sum also belongs to M. Denote by $\mathfrak{m}$ the subspace spanned by $\{y_{ij}, \ (i,j) \in M\}$. By definition of M, the subspace $\mathfrak{m}$ is an ideal in the Lie algebra $\mathfrak{n}$.

Denote by $\mathcal{L}$ the factor algebra $\mathfrak{n}/\mathfrak{m}$ and by L the corresponding factor group of N with respect to normal subgroup $\exp(\mathfrak{m})$. Note that the conjugate space $\mathcal{L}^*$ is a subspace in $\mathfrak{n}^*$ which consists of $f \in \mathfrak{n}^*$ that annihilate $\mathfrak{m}$. The coadjoint L-orbit for $f \in \mathcal{L}^*$ coincides with its N-orbit.

1. Method of construction of the diagram

As we say above in the paper [9] we correspond to the Lie algebra $\mathcal{L}$ a diagram $\mathcal{D}_\mathcal{L}$. Let us state the method of construction of the diagram $\mathcal{D}_\mathcal{L}$ and formulate the main assertions of the paper [9]. Consider the order $\succ$ on the set Δ_+ such that

$$(n,1) \succ (n-1,1) \succ \ldots \succ (2,1) \succ (n,2) \succ \ldots \succ (3,2) \succ \ldots \succ (n,n-1).$$

By the ideal $\mathfrak{m}$ we construct the diagram that is a $n \times n$-matrix, in which all places (i,j), $i \leq j$, are not filled and all other places (i.e. places in Δ_+) are filled by the the symbols "$\otimes$", "$\bullet$", "$+$" and "$-$" according to the following rules. The places $(i,j) \in M$ are filled by the symbol "$\bullet$". Let us say that this procedure is the zero step in construction of the diagram.

We put the symbol "$\otimes$" on the greatest(in the sense of order $\succ$) place in $\Delta_+ \setminus M$. Note that this symbol will take place in the first column, if the set of pairs of the form $(i,1)$ in $\Delta_+ \setminus M$ is not empty. Suppose that we put the symbol "$\otimes$" on the place (k,t), $k > t$. Further, on all places (k,a), $t < a < k$,

we put the symbol "$-$" and on all places (b,t), $1<b<k$, we put the symbol "$+$". This procedure finishes the first step of construction of diagram.

Further, we put the symbol "$\otimes$" on the greatest (in the sense of order $\succ$) empty place in Δ_+. As above we put the symbols "$+$" "$-$" on the empty places taking into account the following: we put the symbols "$+$" and "$-$" in pairs; if the both places (k,a) and (a,t), where $k>a>t$ are empty, we put "$-$" on the first place and "$+$" on the second place; if one of these places (k,a) or (a,t) are already filled, then we do not fill the other place. After this procedure we finish the step that we call a second step.

Continuing the procedure further we have got the diagram. We denote this diagram by $\mathcal{D}_\mathcal{L}$. The number of last step is equal to the number of symbols "$\otimes$" in the diagram.

Example 1. Let $n=7$, $\mathfrak{m}=Ky_{51}\oplus Ky_{61}\oplus Ky_{71}\oplus Ky_{62}$. The corresponding diagram is as follows

$$\mathcal{D}_\mathcal{L}=\begin{array}{|c|c|c|c|c|c|c|}\hline & & & & & & \\ \hline + & & & & & & \\ \hline + & + & & & & & \\ \hline \otimes & - & - & & & & \\ \hline \bullet & + & + & \otimes & & & \\ \hline \bullet & \otimes & - & + & - & & \\ \hline \bullet & \bullet & \otimes & \otimes & - & - & \\ \hline \end{array}$$

We construct the diagram in 5 steps, beginning with zero step:

$$\begin{array}{|c|c|c|c|c|c|c|}\hline & & & & & & \\ \hline & & & & & & \\ \hline & & & & & & \\ \hline & & & & & & \\ \hline \bullet & & & & & & \\ \hline \bullet & & & & & & \\ \hline \bullet & \bullet & & & & & \\ \hline \end{array} \Rightarrow \begin{array}{|c|c|c|c|c|c|c|}\hline & & & & & & \\ \hline + & & & & & & \\ \hline + & & & & & & \\ \hline \otimes & - & - & & & & \\ \hline \bullet & & & & & & \\ \hline \bullet & & & & & & \\ \hline \bullet & \bullet & & & & & \\ \hline \end{array} \Rightarrow \begin{array}{|c|c|c|c|c|c|c|}\hline & & & & & & \\ \hline + & & & & & & \\ \hline + & + & & & & & \\ \hline \otimes & - & - & & & & \\ \hline \bullet & + & & & & & \\ \hline \bullet & \otimes & - & & - & & \\ \hline \bullet & \bullet & & & & & \\ \hline \end{array} \Rightarrow$$

$$\begin{array}{|c|c|c|c|c|c|c|}\hline & & & & & & \\ \hline + & & & & & & \\ \hline + & + & & & & & \\ \hline \otimes & - & - & & & & \\ \hline \bullet & + & + & & & & \\ \hline \bullet & \otimes & - & & - & & \\ \hline \bullet & \bullet & \otimes & & - & & \\ \hline \end{array} \Rightarrow \begin{array}{|c|c|c|c|c|c|c|}\hline & & & & & & \\ \hline + & & & & & & \\ \hline + & + & & & & & \\ \hline \otimes & - & - & & & & \\ \hline \bullet & + & + & & & & \\ \hline \bullet & \otimes & - & + & - & & \\ \hline \bullet & \bullet & \otimes & \otimes & - & - & \\ \hline \end{array} \Rightarrow \begin{array}{|c|c|c|c|c|c|c|}\hline & & & & & & \\ \hline + & & & & & & \\ \hline + & + & & & & & \\ \hline \otimes & - & - & & & & \\ \hline \bullet & + & + & \otimes & & & \\ \hline \bullet & \otimes & - & + & - & & \\ \hline \bullet & \bullet & \otimes & \otimes & - & - & \\ \hline \end{array}$$

Denote by S (resp. C^+, C^-) the set of pairs (i,j), filled in the diagram by the symbol "$\otimes$" (resp. "$+$", "$-$"). The set Δ_+ of positive roots decomposes into a union of disjoint subsets: $\Delta_+=M\sqcup C^+\sqcup C^-\sqcup S$.

Denote by $\mathbb{A}_m$ the Poisson algebra $K[p_1, \ldots, p_m; q_1, \ldots, q_m]$ with the bracket $\{p_i, q_j\} = \delta_{ij}$.

Recall that a Poisson algebra $\mathcal{A}$ is a tensor product of two Poisson algebras $\mathcal{B}_1 \otimes \mathcal{B}_2$, if $\mathcal{A}$ is isomorphic to $\mathcal{B}_1 \otimes \mathcal{B}_2$ as commutative associative algebra and $\{\mathcal{B}_1, \mathcal{B}_2\} = 0$.

The next theorems 1.1 and 1.2 are the main results of the paper [9].

Theorem 1.1 [7]. *There exist $z_1, \ldots, z_s \in K[\mathcal{L}^*]^L$, where $s = |S|$ such that*
1) any $z_i = y_{\xi_i} Q_i + P_{>i}$, where Q_i is some product of powers of $z_1, \ldots, z_{i-1}$ and $P_{>i}$ is a polynomial in $\{y_\eta\}$, $\eta \succ \xi_i$;
2) denote by $\mathcal{Z}$ the set of denominators generated by $z_1, \ldots, z_s$; the localization $K[\mathcal{L}^*]_{\mathcal{Z}}$ of the algebra $K[\mathcal{L}^*]$ with respect to the set of denominators $\mathcal{Z}$ is isomorphic as a Poisson algebra to the tensor product $K[z_1^{\pm}, \ldots, z_s^{\pm}] \otimes \mathbb{A}_m$ for some m.

Theorem 1.1 directly implies the following

Theorem 1.2 [7].
1) The field of invariants $K(\mathcal{L}^*)^L$ coincides with the field $K(z_1, \ldots, z_s)$.
2) The maximal dimension of a coadjoint orbit in $\mathcal{L}^*$ equals to the number of symbols "+" "−" in the diagram $\mathcal{D}_{\mathcal{L}}$.
3) Index of Lie algebra $\mathcal{L}$ coincides with the number of symbols "$\otimes$" in the diagram.

We have to remark that the elements $\{z_i\}$ are constructed by induction in the proof of theorem 1.1; this leaves unsolved the question of finding the exact formula for these elements (for instance, in terms of coefficients of characteristic matrix). We return to the question of an exact formula for generators of the field of invariants in the sequel section of this paper.

Let us formulate two auxiliary statements from [9], that we shall use in this paper. Denote by B_i the set of pairs (a, b), $a > b$ that remains unfilled after the ith step in the procedure of filling the diagram. The subsets B_i form the chain:
$$B_0 \supset B_1 \supset \ldots \supset B_s = \emptyset,$$
where $s = |S|$. Denote $A_i = B_i \sqcup M$, $\mathfrak{n}_i = \mathrm{span}\{y_\eta : \ \eta \in A_i\}$, $\mathcal{L}_i = \mathfrak{n}_i/\mathfrak{m}$. Here $\Delta_+ = A_0$.

Let $S = \{\xi_1 \succ \ldots \succ \xi_s\}$. Recall that the place $\xi_i \in S$ is filled by the symbol "$\otimes$" at the ith step.

For $1 \leq i \leq s$ we denote by C_i^- (resp. C_i^+) the subset of pairs (a, b), $a > b$, that are filled by the symbol "−" (resp."+") at the ith step.

Proposition 1.3 [9, Lemma 1]. The subspace $\mathfrak{n}_i$ (resp. $\mathcal{L}_i$) in $\mathfrak{n}$ (resp. $\mathcal{L}$) is a Lie subalgebra.

For any $1 \leq i \leq s$ we denote

$$D_i^- = \{\eta \in \Delta_+ | \ \xi_i \succ \eta \quad \text{and} \quad \eta \in \bigsqcup_{1 \leq j \leq i} C_j^-\}.$$

$$D_i^+ = \{\eta \in \Delta_+ | \ \xi_i \succ \eta \quad \text{and} \quad \eta \in \bigsqcup_{1 \leq j \leq i} C_j^+\}.$$

Remark that the places $\eta \in D_i^+$ are situated in the same column as ξ_i.

By $\mathfrak{d}_i^-$, where $1 \leq i \leq s$, we denote a linear subspace in $\mathfrak{n}$, spanned by the vectors y_η, where $\eta \in D_i^-$.

Proposition 1.4 [9, Lemma 2]. For any $1 \leq i \leq s$ the subspace $\mathfrak{d}_i^-$ is a Lie subalgebra in $\mathfrak{n}$.

In the example 1 we have

$$\xi_1 = (4,1), \quad \xi_2 = (6,2), \quad \xi_3 = (7,3), \quad \xi_4 = (7,4), \quad \xi_5 = (5,4);$$

$$\begin{array}{ll}
C_1^- = \{(4,2),(4,3)\}, & \mathfrak{d}_1^- = \mathrm{span}\{y_{42}, y_{43}\}; \\
C_2^- = \{(6,3),(6,5)\}, & \mathfrak{d}_2^- = \mathrm{span}\{y_{42}, y_{43}, y_{63}, y_{65}\}; \\
C_3^- = \{(7,5)\}, & \mathfrak{d}_3^- = \mathrm{span}\{y_{43}, y_{63}, y_{65}, y_{75}\}; \\
C_4^- = \{(7,6)\}, & \mathfrak{d}_4^- = \mathrm{span}\{y_{65}, y_{75}, y_{76}\}; \\
C_5^- = \emptyset, & \mathfrak{d}_5^- = \mathfrak{d}_4^-.
\end{array}$$

2. Associated permutation and its decompositions

We correspond to a Lie algebra $\mathcal{L}$ the permutation defined as follows.

Definition 2.1. Denote by $w = w_{\mathcal{L}}$ the permutation of S_n such that

1) $w(1) = \max\{1 \leq i \leq n | \ (i,1) \notin M\}$;

2) $w(t) = \max\{1 \leq i \leq n | \ (i,t) \notin M, \quad i \notin \{w(1), \ldots, w(t-1)\}\}$ for all $2 \leq t \leq n$.

As usual we denote by $l(w)$ the number of multipliers in decomposition of w into products of simple reflections. The number $l(w)$ coincides with the number of inversions in the rearrangement $(w(1), \ldots, w(n))$.

Theorem 2.2. *The number $l(w)$ coincides with* $\dim \mathcal{L}$.

Proof. Denote by $l(w)^{(t)}$ the number of k, that $k > t$ and $w(k) < w(t)$. Show that $l(w)^{(t)}$ coincides with $\dim \mathcal{L}^{(t)}$, where $\mathcal{L}^{(t)} = \mathrm{span}\{y_{it} | \ (i,t) \notin M\}$.

Let $a^{(t)}$ be the greatest number such that the pair $(a^{(t)}, t)$ do not lie in M. Obviously, $a^{(t)} \geq t$ and $\dim \mathcal{L}^{(t)} = a^{(t)} - t$.

On the other hand, the set $\{w(1), \ldots, w(t)\}$ contains in the segment $[1, a^{(t)}]$. According to definition of w, any c of the segment $[1, a^{(t)}]$, that do not lie in

$\{w(1), \ldots, w(t)\}$, has the form $c = w(k)$, for some $k > t$ such that $w(k) < w(t)$. The segment $[1, a^{(t)}]$ decomposes:

$$[1, a^{(t)}] = \{w(1), \ldots, w(t)\} \bigsqcup \{w(k) | \ k > t, \ w(k) < w(t)\}.$$

Hence, $l(w)^{(t)} = a^{(t)} - t = \dim \mathcal{L}^{(t)}$. Since

$$l(w) = \sum_{t=1}^{n} l(w)^{(t)} \quad \text{and} \quad \mathcal{L} = \oplus_{t=1}^{n} \mathcal{L}^{(t)},$$

then $l(w) = \dim \mathcal{L}$. $\Box$

Recall that $S = \{\xi_1 \succ \xi_2 \succ \ldots \succ \xi_s\}$. Put $w_0 = 1$. For any $1 \leq i \leq s$ we denote

$$w_i = r_{\xi_1} r_{\xi_2} \cdots r_{\xi_i}, \tag{1}$$

The set $\{\eta | \ \xi_i \succ \eta\}$ decomposes into a disjoint union

$$B_i \sqcup D_i^- \sqcup D_i^+.$$

Proposition 2.3.
1) If $\eta \in B_i$, then $w_i(\eta) > 0$.
2) If $\eta \in D_i^- \cup D_i^+$, then $w_i(\eta) < 0$.
Proof will be proceeded by induction on $0 \leq i \leq s$. For $i = 0$ the statement is evident. Suppose that the statement is true for all numbers that is less than i. Let us prove the statement for i.

Let $\xi_i = (k, t), \ \ k > t$. Decompose $\{\eta | \ \xi_i \succ \eta\}$ into four subsets $I \sqcup II \sqcup III \sqcup IV$, where

$$I = \{(b, t) | \ t < b < k\},$$

$$II = \{(k, c) | \ t < c < k\},$$

$$III = \{(b, k) | \ k < b\},$$

and IV is consists of pairs $\{\eta | \ \xi_i \succ \eta\}$, that do not lie in I, II and III.

If $\eta \in IV$, then $r_{\xi_i}(\eta) = \eta$, and the statement of proposition follows from the induction assumption.

Case 1. Let $\eta \in I \cap B_i$. Then $\eta = (b, t)$, where $t < b < k$, and the place η is empty after the ith step. Therefore, the place (k, b) is filled (by the symbol "$-$") before the ith step (otherwise at the ith step the place (b, t) is filled by the symbol "$+$") . The induction assumption implies that $w_{i-1}(\eta') < 0$ for $\eta' = (k, b)$. Hence,

$$w_i(\eta) = w_{i-1} r_{\xi_i}(\eta) = -w_{i-1}(\eta') > 0.$$

Case 2. Let $\eta \in I \cap (D_i^- \cup D_i^+)$. In the following items 2a) and 2b) we shall show that $w_i(\eta) < 0$.
2a). Let $\eta \in I \cap C_i^+$. Then $\eta = (b, t)$, where $t < b < k$, and the place η is filled by the symbol "+" at the ith step. Hence, the place (k, b) is filled by the symbol "−" at the ith step and empty at the previous $(i-1)$th step. the induction assumption implies $w_{i-1}(\eta') > 0$ for $\eta' = (k, b)$. Therefore,

$$w_i(\eta) = w_{i-1} r_{\xi_i}(\eta) = -w_{i-1}(\eta') < 0.$$

2b). Let $\eta \in I \cap (D_i^- \cup D_i^+)$ and $\eta \notin C_i^+$. In this case the place η is filled by one of the symbols "+" or "−" at some jth step, $j < i$.

Let us show that the place η', that is equal to $-r_{\xi_i}(\eta) = (k, b)$, is free after the ith step (i.e. $\eta' \in B_i$). Let us suppose the contrary. Then $\eta' \in C_j^-$ for some $j < i$. The place ξ_j is situated in a column with the number less or equal to t. Since $\eta \in D_i^- \cup D_i^+$, then $\eta \in D_i^-$ or $\eta \in D_i^+$. From $\eta \in D_i^-$ and $\eta' \in C_j^- \subset D_i^-$ we have got $\xi_i = \eta + \eta' \in D_i^-$ (see proposition 1.4). This leads to a contradiction.

We have to treat the case $\eta \in D_i^+$. Then $\eta \in C_m^+$ for some $\xi_m = (c, t),\ c > k$, that situates in the same the tth columns as ξ_i, but below ξ_i. Then at the mth step the place η, that is equal to (b, t), will be filled by the symbol "+", and the place (c, b) — by the symbol "−". At the mth step the place (c, k) is already filled by the symbol "−" (otherwise at the mth step (c, k) will be filled by "−", and ξ_i, that is equal to (k, t), will be filled by "+"). Finally, after the $(m-1)$th step the places (k, b), that equals to η', and (c, k) are already filled by the symbol "−", and at the same time the place (c, b) is empty. This contradicts to the statement that $\mathfrak{d}_{m-1}^-$ is a subalgebra (see proposition 1.4).

At last, $\eta' \in B_i$. Then $w_{i-1}(\eta') > 0$ and, therefore, $w_i(\eta) = -w_{i-1}(\eta') < 0$. This proves 2) for the case $\eta \in I$.
Case 3. $\eta \in II \cap B_i$. The case is treated similarly to the case 1.
Case 4. $\eta \in II \cap (D_i^- \cup D_i^+)$. The case is treated similarly to the case 2.
Case 5. $\eta \in III$. As in the proof of theorem 2.2, we denote by $a^{(t)}$ the greatest number, that depends on t, such that $(a^{(t)}, t) \notin M$. Then all pairs (c, t), where $c > t$, lie in M. Recall that $III = \{(b, k) |\ k < b\}$. Decompose III into two subsets :
$III_1 = \{(b, k) \in III |\ k < b \le a^{(t)}\}$,
$III_2 = \{(b, k) \in III |\ n \ge b > a^{(t)}\}$.
5a. Let $\eta \in III_2$. By definition of $a^{(t)}$, the rectangle $(a^{(t)}, n] \times [1, t]$ is filled by the symbol "•" in the diagram $\mathcal{D}_{\mathcal{L}}$. All places $\xi_j,\ \ j \le i$, are situated upper this rectangle. Hence $III_2 \subset B_i$ and $w_i(\eta) > 0$. This proves the statement of

proposition in this case.

5b. Let us show that $III_1 \subset \bigcup_{j<i} C_j^-$. Suppose the contrary, let there exists $\eta = (b,k) \in III_1$ such that

$$\eta \notin \bigcup_{j<i} C_j^-.$$

Then the place η is empty after the ith step. Consider the place (b,t). Since $b > k$, then the place (b,t) is filled before the ith step by one of the symbols "$\otimes$", "$-$" or "$+$".

On the other hand, the symbol "$\times$" can not take the place (b,t), since then $\eta = \xi_m$ for some $m < i$. At the mth step the place $\xi_i = (k,t)$ will be filled by the symbol "$+$" (resp. (b,k) — by the symbol "$-$"). The place (b,t) can not be filled by the symbol, since in this case $(k,t) \in B_{i-1}$, $(b,k) \in B_i \subset B_{i-1}$ and $(b,t) \notin B_{i-1}$, this contradicts to proposition 1.3 (the subspace $\mathfrak{n}_{i-1}$ is not a subalgebra).

Finally, suppose that the place (b,t) is filled by the symbol "$+$". Then there exists $\xi_j = (c,t) \in S$, $j < i$, $c > b$, such that at the jth step the symbol "$+$" appears on the place (b,t) and the symbol "$-$" on the place (c,b). At the same jth step the place (c,k) must be already filled, otherwise (k,t) would be filled by the symbol "$+$" after the ith step. According to the procedure of arrangement of symbols, the place (c,k) may be filled only by the symbol "$-$". Thus, at the jth step we have got that the places (c,b) and (b,k) are empty, and the place (c,k) is filled by the symbol "$-$". this contradicts to the statement that $\mathfrak{n}_{j-1}$ is a subalgebra (see proposition 1.4). The statement of the item 5b is proved.

5c. Let us show that $w_i(\eta) < 0$ for any $\eta = (b,k) \in III_1$. Let $\eta = (b,k)$. Then $(b,t) = r_{\xi_i}(b,k)$. Let m be the greatest number such that $\xi_m \succeq (b,t)$. The following two cases is possible.

a) $m = i-1$. Then either $(b,t) = \xi_{i-1}$, or $(b,t) \in D_{i-1}^+ \sqcup D_{i-1}^-$. In any case $w_{i-1}(b,t) < 0$. We have got

$$w_i(\eta) = w_{i-1} r_{\xi_i}(b,k)) = w_{i-1}(b,t)) < 0.$$

b) $m > i-1$. Let $\xi_{i-1} = (c,t)$. Then $b > c$ and

$$r_{\xi_{i-1}} r_{\xi_i}(b,k) = r_{\xi_{i-1}}(b,t) = (b,c).$$

Note that $r_{\xi_p}(b,c) = (b,c)$ for all $m < p < i-1$. Hence,

$$w_i(\eta) = w_m r_{\xi_{m+1}} \dots r_{\xi_{i-1}} r_{\xi_i}(b,k) = w_m(b,c). \tag{2}$$

We have to prove that $w_m(b,c) < 0$.

Since in the pair ξ_m the number of column is less or equal to t, and $c > t$, then after the mth step the place (b, c) is either empty, or filled by the symbol "$-$". By induction assumption, the inequality $w_m(b, c) < 0$ is equivalent to the statement that the place (b, c) is filled after the mth step (by the symbol "$-$").

Suppose that the place (b, c) is empty after the mth step. The place (b, t) is filled after the ith step, as long as $b > k$. By definition of m, the place (b, t) is filled after the mth step (by one pf the symbols $\otimes$, "$+$", "$-$"). So, after the mth step the place (b, t) is already filled, and the places (b, c) and (c, t) are empty. This contradicts to the statement that $\mathfrak{n}_m$ is a subalgebra. Thus, (b, c) is already filled after the mth step by the symbol "$-$" and, therefore, $w_m(b, c) < 0$. This proves that $w_i(\eta) < 0$. $\square$

Corollary 2.4. *If $\eta \in B_i$, then $w_j(\eta) > 0$ for any $1 \leq j \leq i$.*

Proof follows from the inclusion $B_i \subset B_j$. $\square$

Denote by $w^{(t)}$ the product of reflections r_ξ (arranged in the decreasing order in the sense of $\succ$), where the number of column of ξ is equal to t. We construct the system of permutations

$$w^{[t]} = w^{(1)} \ldots w^{(t)}. \tag{2}$$

Remark that $w^{[t]}$ coincides with w_i, where ξ_i is the least (in the sense of $\succ$) root among all roots lying in the first t columns.

Let as above $a^{(t)} = \max\{c| \ (c, t) \notin M\}$.

Proposition 2.5. *We claim that*

1) $w^{[t]}(\eta) > 0$ *for any* $\eta = (b, t), \quad a^{(t)} < b \leq n.$

2) $w^{[t]}(\eta) < 0$ *for any* $\eta = (b, t), \quad t < b \leq a^{(t)}.$

Proof. By definition of $a^{(t)}$, the rectangle $(a^{(t)}, n] \times [1, t]$ is filled by "$\bullet$" in the diagram $\mathcal{D}_{\mathcal{L}}$. All places $\xi_j, \ j \leq i$, are situated upper this rectangle. This implies the statement 1).

We shall prove the statement 2) in each of these cases separately.

i) The tth column does not contain the root of S. In this case all column is filled by the symbol "$-$". We have got $w^{[t]}(\eta) = w^{[t-1]}(\eta)$. By proposition 2.3, we obtain $w^{[t-1]}(\eta) < 0$.

ii) Let (b, t) situate upper all roots from S, lying in the tth column. Then (b, t) is filled by the symbol "$+$" or "$-$". By proposition 2.3, $w^{[t]}(\eta) < 0$.

iii) (b, t) situates lower ξ_i, that is the least in the sense of $\succ$ root of the tth column, or coincides with it.

If $\eta = \xi_i$, then
$$w^{[t]}(\eta) = w_i(\xi_i) = -w_{i-1}(\xi_i) < 0.$$

Let $\xi_i = (k, t)$ and $b > k$. Then $r_{\xi_i}(b, t) = (b, k)$. Let ξ_m be the least root

of S, that is greater (in the sense of $\succ$) or equal to (b,t). By item 5b (see the proof of proposition 2.3), (b,k) is filled by the symbol "$-$" before or during the mth step. Therefore, $w_m(b,k) < 0$. Since $r_{\xi_p}(b,k) = (b,k)$ for any $m < p < i$, then

$$w^{[t]}(\eta) = w_i(\eta) = w_m r_{\xi_{m+1}} \dots r_{\xi_i}(b,t) = w_m r_{\xi_{m+1}} \dots r_{\xi_{i-1}}(b,k) = w_m(b,k) < 0.$$

$\square$

Let as above $S = \xi_1 \succ \ldots \succ \xi_s\}$. Recall that $(k,t) \in S$ if and only if the place (k,t) is filled by the symbol "$\otimes$" in the diagram $\mathcal{D}_{\mathcal{L}}$.

Theorem 2.6. *We claim that* $w = r_{\xi_1} r_{\xi_2} \cdots r_{\xi_s}$.

Proof. according to formula (2)

$$w^{[n]} = r_{\xi_1} r_{\xi_2} \cdots r_{\xi_s}.$$

Let us prove $w^{[n]}(t) = w(t)$ for any $1 \leq t \leq n$, using induction on t. For $t = 1$ the statement is evident. Suppose that the statement is true for the numbers less than t. Now we shall prove it for t.

By definition, $w(t)$ is a greatest number among the numbers of the segment $[1,n]$ of positive integers that do not lie in

$$\Lambda_t = \{w(1), \ldots, w(t-1)\} \sqcup (a^{[t]}, n]. \tag{3}$$

By induction assumption, $w(j) = w^{[n]}(j)$ for any $1 \leq j \leq t-1$. Note that $w^{[n]}(j) = w^{[t]}(j)$, as long as $r_\xi(j) = j$ for all $\xi \in S$, lying in the columns with numbers greater than t. Hence, $w(j) = w^{[t]}(j)$ for all $1 \leq j \leq t-1$. This gives a possibility to substitute w for $w^{[t]}$ in (3).

By definition of $a^{(t)}$, the rectangle $(a^{(t)}, n] \times [1,t]$ is filled by the symbol "$\bullet$" in the diagram $\mathcal{D}_{\mathcal{L}}$. The places ξ_j, $j \leq i$ situates upper this rectangle. Hence, for any $p \in (a^{[t]}, n]$ we have $w^{[t]}(p) = p$.

The set Λ_t may be represented in the form

$$\Lambda_t = \{w^{[t]}(1), \ldots, w^{[t]}(t-1)\} \sqcup \{w^{[t]}(p) |\ a^{[t]} < p \leq n\}. \tag{4}$$

All elements of the segment $[1,n]$ of positive integers, that do not lie in Λ_t, have the form $w^{[t]}(k)$, where $t \leq k \leq a^{(t)}$.

From item 2) of proposition 2.5, $w^{[t]}(t) > w^{[t]}(k)$ for any $t < k \leq a^{(t)}$. Therefore, $w^{[t]}(t)$ is a greatest number among all numbers of the segment $[1,n]$ of positive integers, that do not lie in Λ_t. We conclude that $w^{[t]}(t) = w(t)$. Finally, $w^{[t]}(t) = w^{[n]}(t)$, as long as $r_\xi(t) = t$ for all $\xi \in S$, lying in columns with numbers greater than t. At last, $w^{[n]}(t) = w(t)$. $\square$

Denote by $A^{(t)}$ the set $\eta \in A$, that has the form (b,t) for some $b > t$.

Theorem 2.7. *Let $\eta \in A^{(t)}$, then*
1) the place η is filled in the diagram $\mathcal{D}_\mathcal{L}$ by the symbol "$-$" iff $w^{[t-1]}(\eta) < 0$;
2) the place η is filled in the diagram $\mathcal{D}_\mathcal{L}$ by the symbol "$\bullet$" iff $w^{[t]}(\eta) > 0$;
3) η is filled in the diagram $\mathcal{D}_\mathcal{L}$ by the symbol "$+$" or "$\otimes$" iff $w^{[t-1]}(\eta) > 0$ $w^{[t]}(\eta) < 0$.

Proof is a corollary of the propositions 2.3 and 2.5. $\square$

3. Field of invariants

In this section we formulate the conjecture on the structure of field of invariants of the coadjoint representation of the Lie algebra $\mathcal{L}$.

To any $\xi \in S$ we shall correspond a polynomial P_ξ. Let $\xi = \xi_m = (k,t) \in S$, where $k > t$. Denote by w_ξ the permutation $w_m = r_{\xi_1} \ldots r_{\xi_m}$.

Case 1. $w_\xi(t) > t$. On can show that in this case $w_\xi(t) = k$. Put

$$J := J(\xi) = \{1 \le j \le t : \; w_\xi(j) \ge w_\xi(t)\}, \qquad I := I(\xi) = wJ(\xi).$$

Case 2. $w_\xi(t) \le t$. The system $J := J(\xi)$ is defined as in (3.1). Denote

$$I_*(\xi) := I_*(\xi) = \{1 \le i \le n : \; i > t, \; w_\xi(i) < w_\xi(t)\},$$

$$I(\xi) = [w_\xi(t), t] \sqcup I_*(\xi).$$

On can show that in both cases $|I(\xi)| = |J(\xi)|$.

As above $\{y_{ij}\}$ is the standard basis in $\mathfrak{n}$. By the diagram $\mathcal{D}_\mathcal{L}$, we construct the matrix $\Phi_\mathcal{L}$, in which the places $\{(i,j) \in \Delta_+ \setminus M\}$ are filled by the corresponding elements $\{y_{ij}\}$ of the standard basis; the other places are filled by zeroes. For instance, for the Lie algebra $\mathcal{L}$ of the example 1, we have the following diagram $\mathcal{D}_\mathcal{L}$ and matrix $\Phi_\mathcal{L}$:

$$\mathcal{D}_\mathcal{L} = \begin{array}{|c|c|c|c|c|c|c|}\hline \; & \; & \; & \; & \; & \; & \; \\ \hline + & & & & & & \\ \hline + & + & & & & & \\ \hline \otimes & - & - & & & & \\ \hline \bullet & + & + & \otimes & & & \\ \hline \bullet & \otimes & - & + & - & & \\ \hline \bullet & \bullet & \otimes & \otimes & - & - & \\ \hline \end{array}, \qquad \Phi_\mathcal{L} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ y_{21} & 0 & 0 & 0 & 0 & 0 & 0 \\ y_{31} & y_{32} & 0 & 0 & 0 & 0 & 0 \\ y_{41} & y_{42} & y_{43} & 0 & 0 & 0 & 0 \\ 0 & y_{52} & y_{53} & y_{54} & 0 & 0 & 0 \\ 0 & y_{62} & y_{63} & y_{64} & y_{65} & 0 & 0 \\ 0 & 0 & y_{73} & y_{74} & y_{75} & y_{76} & 0 \end{pmatrix}$$

Let λ be a variable. Consider the characteristic matrix $\Phi_\mathcal{L} - \lambda E$. Note that any minor of the characteristic matrix is a polynomial in λ with coefficients in $S(\mathcal{L}) = K[\mathcal{L}^*]$.

Let $M_\xi(\lambda)$ be the polynomial of characteristic matrix with the system of columns $J(\xi)$ and the system of row $I(\xi)$, and P_ξ be its highest coefficient.
Conjecture. The field of invariants of the coadjoint representation of the Lie algebra $\mathcal{L}$ is a field of rational functions in $P_\xi,\ \ \xi \in S$.

In the example 1, S consists of four elements $\{\xi_1, \xi_2, \xi_3, \xi_4\}$. Direct calculations shows that $P_{\xi_1} = y_{41},\ \ P_{\xi_2} = y_{62},\ \ P_{\xi_3} = y_{73},\ \ P_{\xi_4} = y_{74}y_{41} + y_{73}y_{31}$.

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